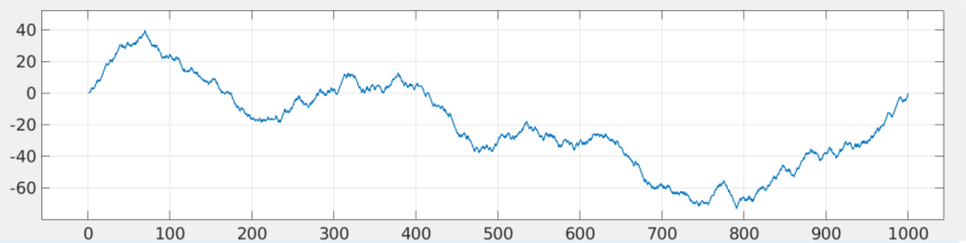
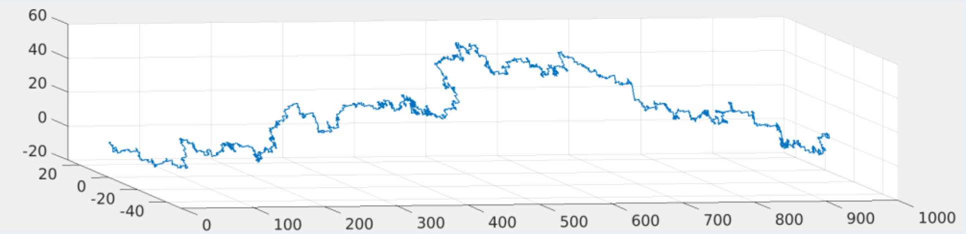
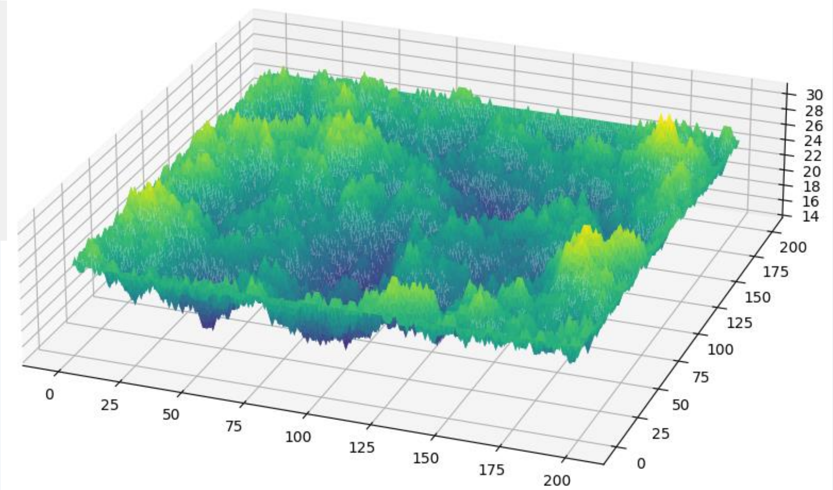
Minimal Surfaces in a Random Environment



1-dim. surface in 2-dim. space (d=1, n=1)



1-dim. surface in 3-dim. space (d=1, n=2)



2-dim. surface in 3-dim. space (d=2, n=1)

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Based on joint works with Barbara Dembin, Dor Elboim and Daniel Hadas,

with Michal Bassan and Shoni Gilboa and with Michal Bassan and Paul Dario

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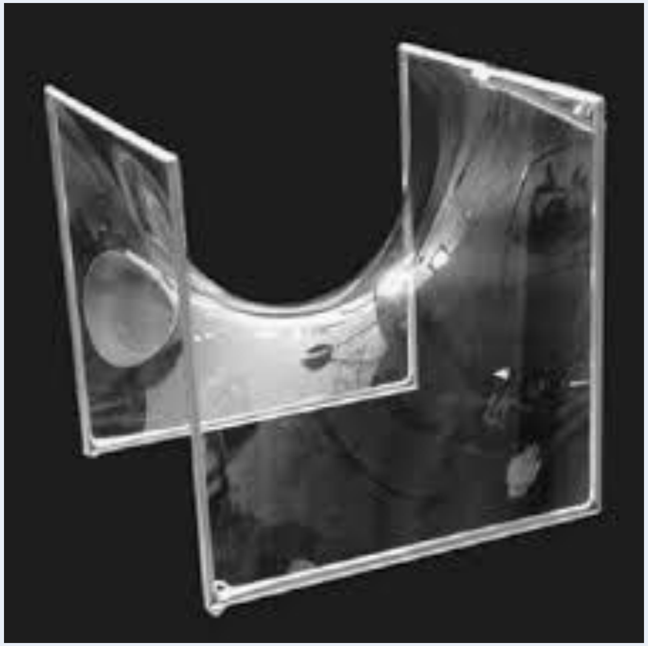
Minimal surfaces

- A minimal surface is a surface that (locally) minimizes the surface area, subject to boundary conditions.
- How would a minimal surface in a random environment look like?
  (e.g., when perturbing the area measure with local random perturbations)



Natural phenomenon

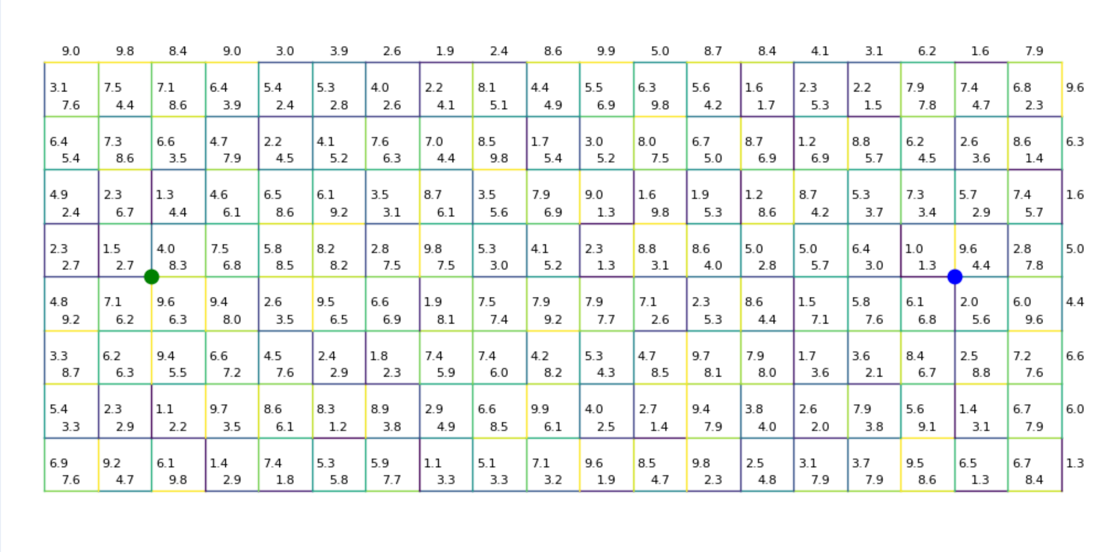
With practical applications

Minimal paths in a random environment: First-passage percolation (Hammersley–Welsh 1965)

- We consider the discrete setting of the lattice $\mathbb{Z}^D$ with $D \geq 2$.
- Edge weights: Independent and identically distributed non-negative $(\tau_e)_{e\in E(\mathbb{Z}^D)}$.
  Distribution of τ_e assumed "nice". For instance, $\tau_e \sim \mathrm{Uniform}[a,b]$ for $b > a > 0$.
- Passage time: A random metric $T_{u,v}$ on $\mathbb{Z}^D$ given by

$$T_{u,v} := \min \sum_{e\in p} \tau_e$$

with the minimum over paths p connecting u and v.



Weights uniform on [1,10]
u = (0,0), v = (15,0)

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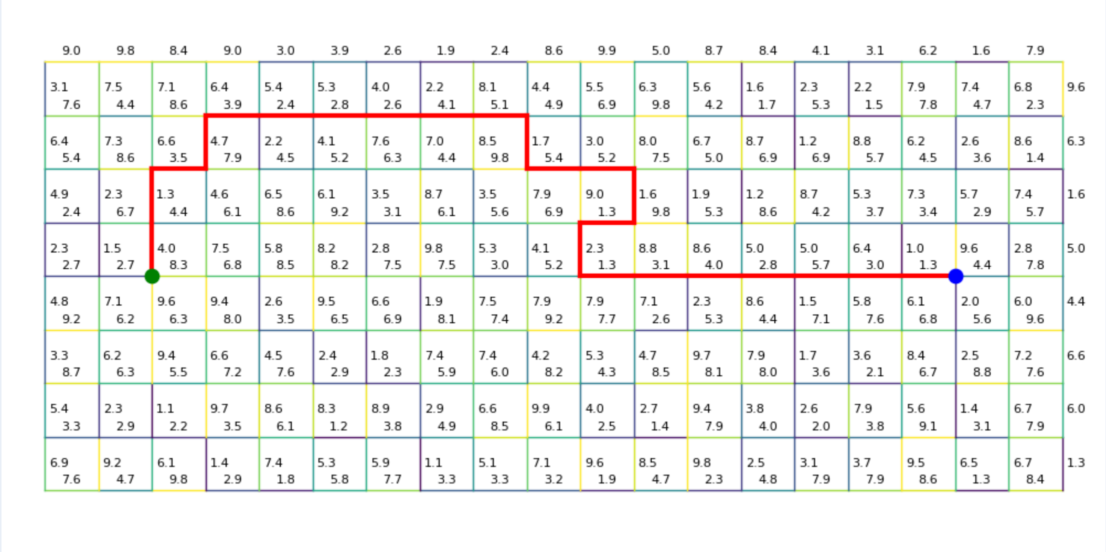
- Geodesic: The unique path p realizing $T_{u,v}$, denoted $\gamma_{u,v}$.
  Geodesic is a 1-dimensional "minimal surface" in D-dimensional space.



Weights uniform on [1,10]
u = (0,0), v = (15,0)
Geodesic in red

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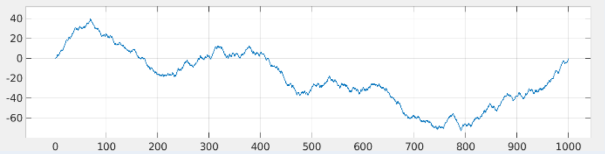
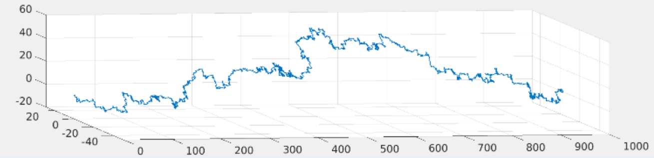
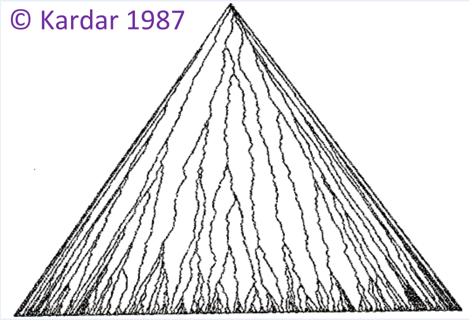
- Geodesic: The unique path p realizing $T_{u,v}$, denoted $\gamma_{u,v}$.
  Geodesic is a 1-dimensional "minimal surface" in D-dimensional space.
- Goal: Understand the large-scale properties of the metric T.
  In particular, understand the geometry and length of long geodesics.



D=2
(dimension d=1,
codimension n=1)



D=3
(dimension d=1,
codimension n=2)

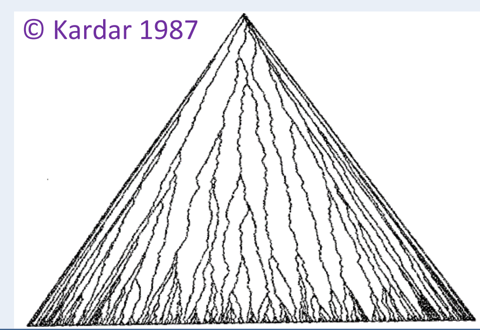


Coalescence of geodesics in D=2 dim.

Roughness and length exponents I

- How far is the geodesic from a straight line? Kardar–Parisi–Zhang (KPZ) universality.
- Roughness exponent: Believed that maximal deviation scales as L^{ξ_D} for length L geodesic, with ξ_D the same in all directions.

- Believed that $\xi_2 = \frac{2}{3}$ (super-diffusivity!)
- Believed that $\xi_{D+1} \leq \xi_D$ and $\xi_D \geq \frac{1}{2}$ for all D.
- Open problem (even) in physics: Does $\xi_D = \frac{1}{2}$ (diffusive behavior) for some D?
- Rigorous results: $\xi_D \geq \frac{1}{D+1}$ for all D (Licea-Newman-Piza 96).
- No non-trivial upper bound on ξ_D!!! (trivial bound $\xi_D \leq 1$)
- Coalescence of geodesics: Non-quantitative results in $D = 2$: Damron-Hanson 15, Ahlberg-Hoffman 16. Strong conditional results for all D: Alexander 20. Quantitative coalescence in $D = 2$ (coalescence exponent $\geq \frac{1}{8}$) and variant of midpoint problem for all D (Dembin-Elboim-P. 24, 25).



Roughness and length exponents II

- Length fluctuation exponent: Believed that the standard deviation of the passage time scales as L^{χ_D}, with χ_D the same in all directions:
$$Std(T_{\mathbf{0},Lv}) = L^{\chi_D + o(1)}$$
- Predicted scaling relation: $\chi_D = 2\xi_D - 1.$
- Rigorous results:

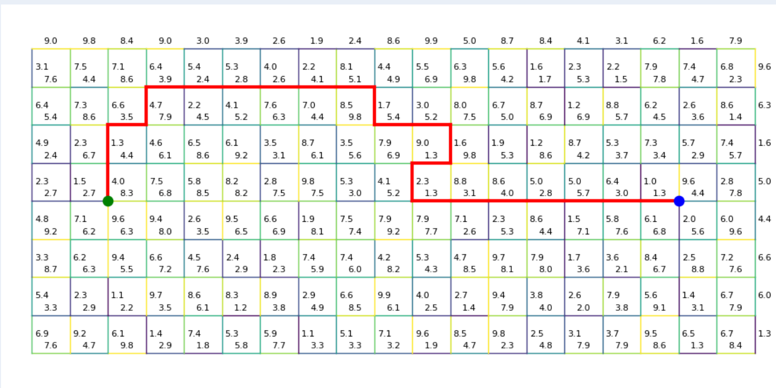
$Std(T_{\mathbf{0},Lv}) \leq c\sqrt{\frac{L}{\log L}}$ (Benjamini-Kalai-Schramm 03)

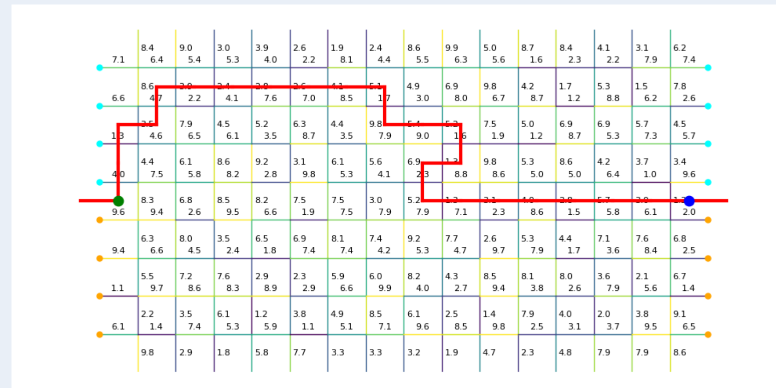
$Std(T_{\mathbf{0},Lv}) \geq c\sqrt{\log L}$ for $D = 2$ (Newman-Piza 95)

- Scaling relation established conditionally (under assumptions which are presently unverified on the exponents and limit shape, Chatterjee 13, Auffinger-Damron 14).
- The book "50 Years of First-Passage Percolation" by Auffinger-Damron-Hanson 15 surveys the rigorous state-of-the-art.
  Many basic questions remain open.
- Detailed understanding available in two dimensions ($\mathrm{D} = 2$) for a related integrable model: Directed last-passage percolation (with specific edge weight distributions). However, no integrable first-passage percolation model is known.

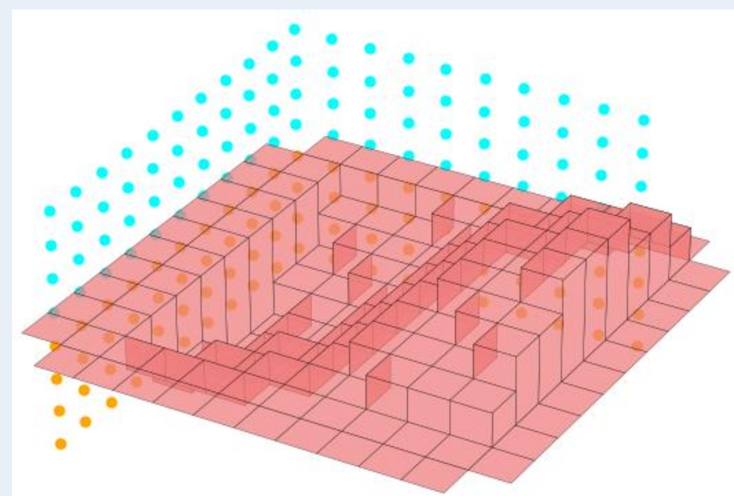
Minimal surfaces in a random environment I

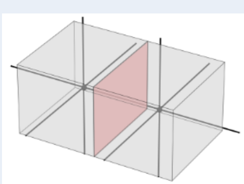
- In dimension $D = 2$, the first-passage percolation geodesic is equivalently represented as the minimal cut in the dual network, which separates the upper half of the boundary from the lower half of the boundary.





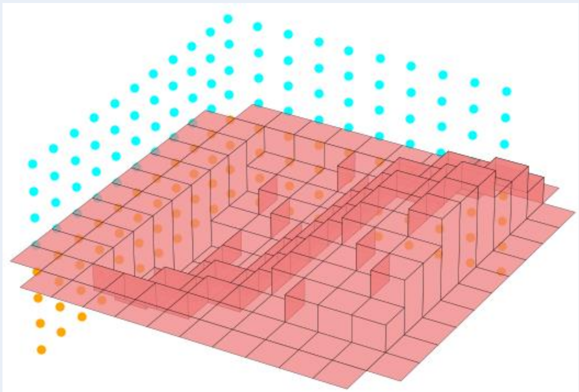
- This point of view extends to higher dimensions D to yield a minimal surface: The surface is composed of the (D-1)-dimensional plaquettes dual to the edges in the minimal cut which separates the upper and lower halves of the boundary.



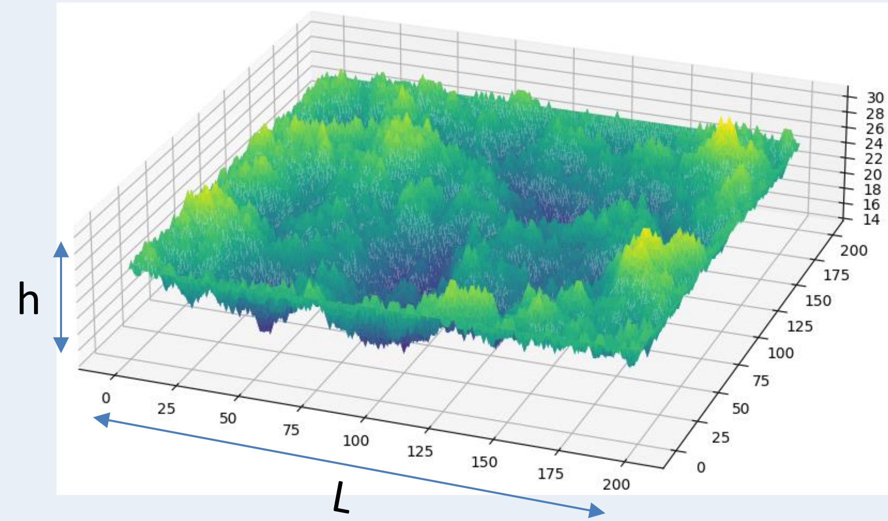


A plaquette is the dual of an edge

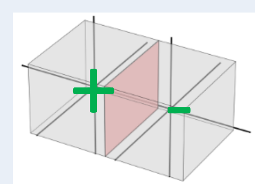
Minimal surfaces in a random environment II



D=3 (dimension d=2, codimension n=1)



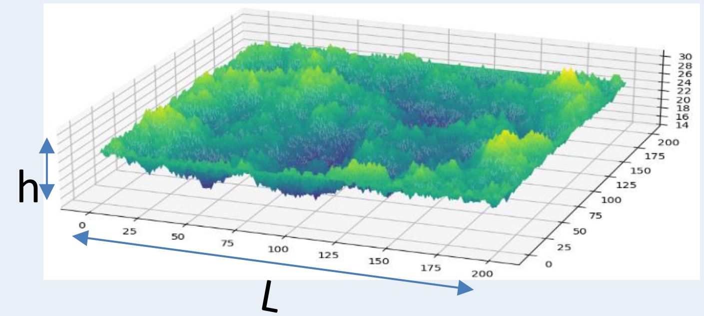
- The minimal surface admits an equivalent statistical mechanics description:
- Recall that $\tau = (\tau_e)_{e\in E(\mathbb{Z}^D)}$ are the edge weights.
- Random-bond Ising model in the environment τ: Configurations $\sigma\colon \mathbb{Z}^D \to \{-1,1\}$ with (quenched, formal) Hamiltonian $H^\tau(\sigma) \coloneqq -\sum_{u\sim v} \tau_{\{u,v\}}\sigma_u\sigma_v$.
- The minimal surface is the domain wall of the ground state of the random-bond Ising model with Dobrushin boundary conditions.



The plaquette is the domain wall between spins -1 and 1

Minimal surfaces in a random environment III

- Basic challenge: How flat is the minimal surface? Does its maximum reach a power of L? Power of logarithm of L? Order 1 in L?



- Weights: Let $b > a > 0$.
  Take the weights (τ_e) independent,
  each distributed as $\text{Uniform}[a, b]$.
- Theorem (Bassan-Dario-P. 25+):
  The surface delocalizes for $d = 2$ (e.g., expected highest sign change above a uniformly chosen vertex in Λ_L is $\geq c\sqrt{\log\log L}$).
- Theorem (Bassan-Gilboa-P. 23): If $\frac{b-a}{a}$ is small then the surface localizes for $d \geq 3$.
- Bovier–Külske 94,96 previously obtained (non-quantitative versions of) such theorems in the disordered Solid-On-Solid approximation (disallowing overhangs).
- Conjecture (Bassan-Gilboa-P. 23. Earlier in physics literature):
  1) $d = 3$: the surface delocalizes when $\frac{b-a}{a}$ is large
  (leading to a roughening transition in the disorder strength in dimension $d = 3$!).
  2) The surface localizes (for all $b > a > 0$) when $d \geq 5$ (possibly also for $d = 4$).

Harmonic minimal surfaces in a random environment

- Minimal surfaces in a random environment (abstract idea):
  d-dimensional surfaces in D=$(d+n)$-dimensional space which minimize the sum of their elastic energy and their environment potential energy, subject to given boundary conditions.
  Of interest in its own right, and related to aforementioned systems.
  We seek a model which is more amenable to analysis!
- Harmonic minimal surfaces in a random environment (Dembin–Elboim–Hadas–P. 24):
  Configurations are $\varphi: \mathbb{Z}^d \to \mathbb{R}^n$ (continuous rather than integer valued!).
  Quenched disorder is $\eta: \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ and disorder strength is $\lambda > 0$.
  In a finite domain $\Lambda \subset \mathbb{Z}^d$, the Hamiltonian is

  $$H^{\eta,\lambda,\Lambda}(\varphi) := \frac{1}{2} \sum_{\substack{u \sim v \\ \{u,v\} \cap \Lambda \neq \emptyset}} \|\varphi_u - \varphi_v\|_2^2 + \lambda \sum_{v \in \Lambda} \eta_{v,\varphi_v}$$

  The minimal surface $\varphi^{\eta,\lambda,\Lambda,\tau}$ is the configuration minimizing $H^{\eta,\lambda,\Lambda}(\varphi)$ among configurations which coincide with boundary conditions $\tau: \mathbb{Z}^d \to \mathbb{R}^n$ outside Λ.
  (an n-component Gaussian free field in a random environment).
- Goal: Study the geometry and energy of the minimal surface on large domains.

Harmonic minimal surfaces in a random environment – explanation of model

- Harmonic minimal surfaces in a random environment (Dembin–Elboim–Hadas–P. 24):
  Configurations are $\varphi\colon \mathbb{Z}^d \to \mathbb{R}^n$ (continuous rather than integer valued!).
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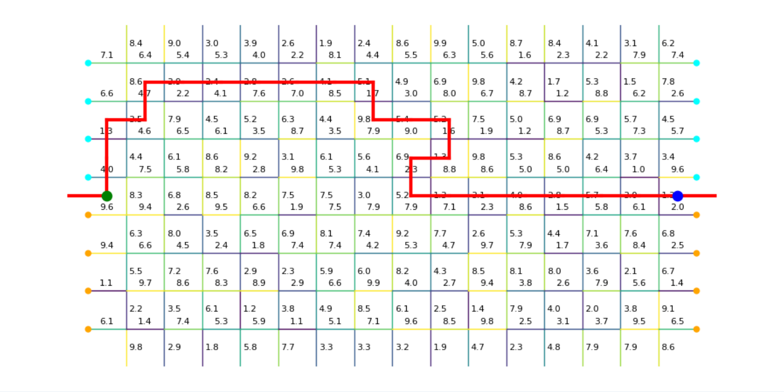
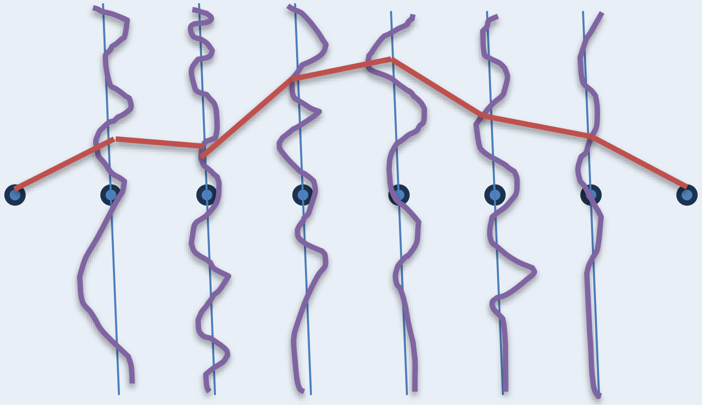


Overhangs allowed
Integer heights
Cost to change height is linear in gradient



No overhangs (function above base)
Real heights (more generally, in $\mathbb{R}^n$)
Cost to change height is quadratic in gradient

Harmonic minimal surfaces in a random environment - background

- Harmonic minimal surfaces in random environment (harmonic MSRE): Configurations are $\varphi: \mathbb{Z}^d \to \mathbb{R}^n$ (continuous rather than integer valued!). Quenched disorder is $\eta: \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ and disorder strength is $\lambda > 0$. In a finite domain $\Lambda \subset \mathbb{Z}^d$, the Hamiltonian is

$$H^{\eta,\lambda,\Lambda}(\varphi) \coloneqq \frac{1}{2} \sum_{\substack{u\sim v \\ \{u,v\}\cap\Lambda\neq\emptyset}} \|\varphi_u - \varphi_v\|_2^2 + \lambda \sum_{v\in\Lambda} \eta_{v,\varphi_v}$$

  The minimal surface $\varphi^{\eta,\lambda,\Lambda,\tau}$ is the configuration minimizing $H^{\eta,\lambda,\Lambda}(\varphi)$ among configurations which coincide with boundary conditions $\tau: \mathbb{Z}^d \to \mathbb{R}^n$ outside Λ.
- Mathematics literature: $d = n = 1$: Bakhtin et al. 16-19 in connection to the Burgers equation. On $\mathbb{R}^{d=1}$, n= 1: Bakhtin–Cator–Khanin 14. Related literature on Brownian polymers in random environment – see review by Comets-Cosco 18. Fixed d and $n \to \infty$: Ben-Arous–Bourgade–McKenna 21 (landscape complexity for the Elastic Manifold, following Fyodorov–Le Doussal 20).
- Physics literature (also related models): Huse–Henley 85, Kardar 87, Natterman 87, Middleton 95, Emig–Nattermann 98, Scheidl–Dincer 00, Le Doussal–Wiese–Chauve 04, Husemann–Wiese 18. Reviews: Forgacs–Lipowsky–Nieuwenhuizen 91 (in Domb–Lebowitz vol. 14), Giamarchi 09, Wiese 22.

Harmonic minimal surfaces in a random environment – disorder

- Our initial focus is on distributions of the disorder $\eta: \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ which are "independent".
- Main example: smoothed white noise, defined as follows:
  - $(\eta_{v,\cdot})_{v \in \mathbb{Z}^d}$ are independent.
  - $\eta_{v,t} = (WN_v * b)(t)$ with WN_v a white noise and b a "bump function" satisfying: (1) $b \geq 0$ and $b(t) = 0$ when $\|t\| \geq 1$, (2) $\int b(t)^2 dt = 1$, (3) b is a Lipschitz function.
- Abstract assumptions (all hold for smoothed white noise):
  - we always assume suitable energy minimizers exist.
  - (stat): for $s: \mathbb{Z}^d \to \mathbb{R}^n$, the shifted disorder $\eta^s_{v,t} := \eta_{v,t-s_v}$ has the same distribution as η.
  - (indep): the $(\eta_{v,\cdot})_{v \in \mathbb{Z}^d}$ are independent. For each v, the process $t \mapsto \eta_{v,t}$ is independent at distance 2.
  - (conc): Write $\mathrm{GE}^{\eta,\lambda,\Lambda,\tau} := H^{\eta,\lambda,\Lambda}(\varphi^{\eta,\lambda,\Lambda,\tau})$ for the ground energy.
    Then for each $\lambda > 0$, $\tau: \mathbb{Z}^d \to \mathbb{R}^n$ and finite $\Delta \subset \Lambda \subset \mathbb{Z}^d$, conditioned on $\eta|_{\Delta^c \times \mathbb{R}^n}$ we have that $\mathrm{Std}(\mathrm{GE}^{\eta,\lambda,\Lambda,\tau}) \leq C\lambda\sqrt{|\Delta|}$ with Gaussian tails on this scale.
- Assumptions (stat)+(indep) allow, e.g., to vary disorder strength between vertices.
- For later reference: (stat)+(conc) hold also for periodic disorder.

Localization and delocalization

- We consider the transversal fluctuations of the harmonic MSRE surface on the domain $\Lambda_L := \{-L, -L+1, \dots, L\}^d$ with zero boundary conditions.
- Theorem (Localization, (stat)+(conc)) (Dembin–Elboim–Hadas–P. 24): There exists $C > 0$, depending only on d, n and the distribution of η, such that for each $v \in \Lambda_L$,

$$\mathbb{E}\left(\left\|\varphi_v^{\eta,\lambda,\Lambda_L}\right\|\right) \le C\sqrt{\lambda}\begin{cases} L^{\frac{4-d}{4}} & d = 1,2,3 \\ \log L & d = 4 \\ 1 & d \ge 5 \end{cases}$$

- Theorem (Delocalization, smoothed white noise) (Dembin–Elboim–Hadas–P. 24): There exists $c > 0$, depending only on the distribution of η and the disorder strength $\lambda > 0$, such that

$$\frac{1}{|\Lambda_L|}\mathbb{E}\left(\left|v \in \Lambda_L : \left\|\varphi_v^{\eta,\lambda,\Lambda_L}\right\| \ge h\right|\right) \ge c$$

with

$$h = \begin{cases} L^{3/5} & d = 1, n = 1 \\ L^{1/2} & d = 1, n \ge 2 \\ L^{\frac{4-d}{4+n}} & d \in \{2,3\} \\ (\log\log L)^{\frac{1}{4+n}} & d = 4 \end{cases}$$

$n = 1$	Lower bound	Predicted	Upper bound
$d = 1$	$L^{0.6}$	$L^{2/3}$	$L^{0.75}$
$d = 2$	$L^{0.4}$	$L^{0.41\pm0.01}$	$L^{0.5}$
$d = 3$	$L^{0.2}$	$L^{0.22\pm0.01}$	$L^{0.25}$
$d = 4$	$(\log\log L)^{0.2}$	$(\log L)^{0.2083\dots}$	$\log L$
$d \ge 5$	1	1	1

- Physics predictions for $n = 1$:
d=1: Huse–Henley 85, Kardar 85, Huse–Henley–D.S.Fisher 85, Kardar–Parisi–Zhang 86,
d=2,3: Middleton 95, Scheidl–Dincer 00, Le Doussal–Wiese–Chauve 04, Husemann–Wiese 18,
d=4: Emig–Nattermann 98,99.

Scaling relation

- Consider now the harmonic MSRE surface on $\Lambda_L = \{-L, -L+1, \dots, L\}^d$ with zero boundary conditions. It is common in the literature to say that the height fluctuations behave as $L^{\xi_{d,n}}$ while the ground energy fluctuations behave as $L^{\chi_{d,n}}$.
- Scaling relation: It is proposed (e.g., Huse–Henley 85) that, at least for $d < 4$,
$$\chi_{d,n} = 2\xi_{d,n} + d - 2$$
We give rigorous versions of this equality for general d, n. Write $\mathrm{Avg}_\Lambda(\cdot)$ for the average operation on Λ. Write $GE^{\eta,\lambda,\Lambda}$ for the energy of the minimal surface.
- Theorem ((stat)+(indep)) (Dembin–Elboim–Hadas–P. 24): There exist $C, c > 0$, depending only on d, such that for all $\mathrm{h} > 0$, all $\lambda > 0$ and unit vector $e \in \mathbb{R}^n$:
First (version of $\chi_{d,n} \geq 2\xi_{d,n} + d - 2$),
$$\mathbb{P}\left(\left|GE^{\eta,\lambda,\Lambda_L} - \mathrm{Med}\left(GE^{\eta,\lambda,\Lambda_L}\right)\right| \geq ch^2 L^{d-2}\right) \geq \frac{1}{3}\mathbb{P}\left(\left|\mathrm{Avg}_{\Lambda_L}\left(\varphi^{\eta,\lambda,\Lambda_L}\right)\cdot e\right| \geq h\right)$$
Second (version of $\chi_{d,n} \leq 2\xi_{d,n} + d - 2$), let $\eta[\Lambda_{\lfloor L/2\rfloor}]$ be η with its middle portion resampled (precisely, $\eta[\Lambda_{\lfloor L/2\rfloor}]$ is obtained by resampling $\eta_{v,\cdot}$ for $v \in \Lambda_{\lfloor L/2\rfloor}$). For $h \geq 1$,
$$\mathbb{P}\left(\left|GE^{\eta,\lambda,\Lambda_L} - GE^{\eta[\Lambda_{\lfloor L/2\rfloor}],\lambda,\Lambda_L}\right| \geq Ch^2 L^{d-2}\right) \leq C\mathbb{P}\left(\max_{v\in\Lambda_L}\left|\varphi_v^{\eta,\lambda,\Lambda_L}\cdot e\right| \geq h\right)$$
Third, for $d = 1$: Define $M_k := \max_{L-k\leq|v|\leq L}\left|\varphi^{\eta,\lambda,\Lambda_L}\cdot e\right|$. Then
$$c \max_{0\leq j\leq\lceil\log_2 L\rceil} 2^{-j}\left(\mathbb{E}M_{2^j}\right)^2 \leq Std\left(GE^{\eta,\lambda,\Lambda_L}\right) \leq C\sum_{0\leq j\leq\lceil\backslash\mathrm{log}_2 L\rceil} 2^{-j}\left(1 + \sqrt{\mathbb{E}M_{2^j}^4}\right)$$

Main identity

- The following deterministic identity is our main tool for analyzing the harmonic MSRE model.
- Fix a finite $\Lambda \subset \mathbb{Z}^d$ and the disorder strength $\lambda > 0$. We abbreviate

$$H^{\eta}(\varphi) \coloneqq H^{\eta,\lambda,\Lambda}(\varphi) = \frac{1}{2} \sum_{\substack{u \sim v \\ \{u,v\} \cap \Lambda \neq \emptyset}} \|\varphi_u - \varphi_v\|_2^2 + \lambda \sum_{v \in \Lambda} \eta_{v,\varphi_v} = \frac{1}{2} \|\nabla \varphi\|_\Lambda^2 + \lambda \sum_{v \in \Lambda} \eta_{v,\varphi_v}$$

- Lemma (Main identity): For each $\varphi: \mathbb{Z}^d \to \mathbb{R}^n$ and $s: \mathbb{Z}^d \to \mathbb{R}^n$ we have

$$H^{\eta^s}(\varphi + s) - H^{\eta}(\varphi) = (\varphi, -\Delta_\Lambda s) + \frac{1}{2} \|\nabla s\|_\Lambda^2$$

where $(-\Delta_\Lambda s)_v \coloneqq \sum_{\substack{u:\, u \sim v \\ \{u,v\} \cap \Lambda \neq \emptyset}} (s_v - s_u)$ and the shifted disorder $\eta^s: \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ is

$$\eta^s_{v,t} \coloneqq \eta_{v,t-s_v}$$

- Proof: Indeed, the disorder term cancels in the first equality of

$$H^{\eta^s}(\varphi + s) - H^{\eta}(\varphi) = \frac{1}{2} (\|\nabla(\varphi + s)\|_\Lambda^2 - \|\nabla \varphi\|_\Lambda^2)$$

$$= \frac{1}{2} \Big(\big(\nabla(\varphi + s), \nabla(\varphi + s)\big)_\Lambda - (\nabla \varphi, \nabla \varphi)_\Lambda \Big) = (\nabla \varphi, \nabla s)_\Lambda + \frac{1}{2} (\nabla s, \nabla s)$$

$$= (\varphi, -\Delta_\Lambda s) + \frac{1}{2} \|\nabla s\|_\Lambda^2$$

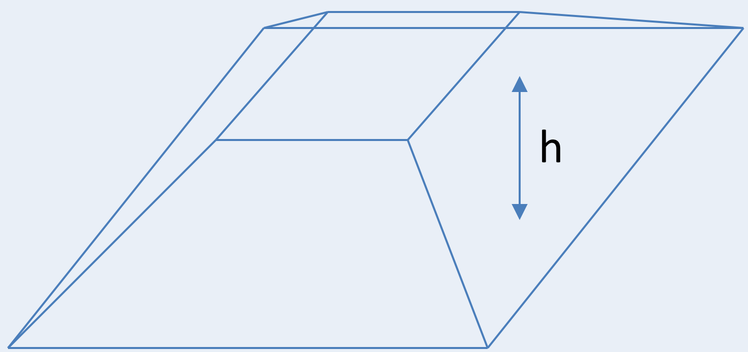
and a discrete Green's identity is used in the last step.

Main identity consequences

- Lemma (Main identity): For each $\varphi: \mathbb{Z}^d \to \mathbb{R}^n$ and $s: \mathbb{Z}^d \to \mathbb{R}^n$ we have
$$H^{\eta^s}(\varphi + s) - H^{\eta}(\varphi) = (\varphi, -\Delta_\Lambda s) + \frac{1}{2}\|\nabla s\|_\Lambda^2$$
where $(-\Delta_\Lambda s)_v := \sum_{\substack{u:\, u\sim v \\ \{u,v\}\cap\Lambda\neq\emptyset}} (s_v - s_u)$ and the shifted disorder $\eta^s: \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ is
$$\eta^s_{v,t} := \eta_{v,t-s_v}$$
- Corollary (Effect of boundary conditions, (stat)): Recall that $\varphi^{\eta,\lambda,\Lambda,\tau}$ is the minimal surface on $\Lambda \subset \mathbb{Z}^d$ with boundary conditions τ. Write $GE^{\eta,\lambda,\Lambda,\tau}$ for its energy. Then
$$\left(\varphi^{\eta,\lambda,\Lambda,\tau}, GE^{\eta,\lambda,\Lambda,\tau}\right) \overset{d}{=} \left(\varphi^{\eta,\lambda,\Lambda,0}, GE^{\eta,\lambda,\Lambda,0}\right) + \left(\bar{\tau}^\Lambda, \frac{1}{2}\left\|\nabla\bar{\tau}^\Lambda\right\|_\Lambda^2\right)$$
where $\bar{\tau}^\Lambda$ is the harmonic extension of τ to Λ.
Extends familiar Shear Invariance to Harmonic Invariance.
- Corollary (Quadratic limit shape, (stat)): expected additional energy for a sloped surface over a flat surface is quadratic in the slope.
- Corollary (Concentration inequality for linear functionals of the surface, (stat)+(conc)): For each $\lambda > 0, \Lambda \subset \mathbb{Z}^d$ finite, $s: \mathbb{Z}^d \to \mathbb{R}^n$ and $r > 0$:
$$\mathbb{P}\left(\left|\left(\varphi^{\eta,\lambda,\Lambda}, -\Delta_\Lambda s\right)\right| \geq r\right) \leq 3 \inf_{\gamma\in\mathbb{R}} \mathbb{P}\left(\left|GE^{\eta,\lambda,\Lambda} - \gamma\right| \geq \frac{r^2}{4\|\nabla s\|_\Lambda^2}\right)$$
- This concentration inequality is used to derive the scaling inequality $\chi_{d,n} \geq 2\xi_{d,n} + d - 2$ and the height fluctuation upper bounds.

Scaling relation intuition

- The quadratic limit shape that the expected energy "cost" for a surface with boundary condition 0 to equal h in the bulk of the domain is $h^2 \mathrm{L}^{d-2}$.
- This cost may be compensated by the energy "gain" from the disorder in the bulk, which (should be like) the typical energy fluctuation L^{χ}.
- Comparing the two expressions shows that the typical height fluctuation $h = L^{\xi}$ satisfies $\chi = 2\xi + d - 2$.



Minimal surfaces in a strongly correlated random environment I

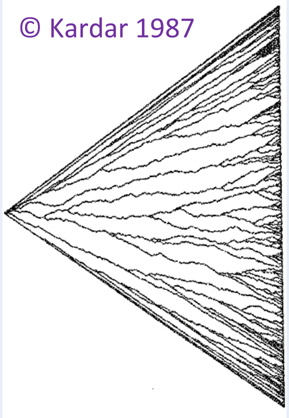
- The lattice minimal surfaces that we saw were modeling the domain walls of the random-bond Ising model.
- It is also of interest to study the domain walls of the random-field Ising model.
- Random-field Ising model: Configurations $\sigma: \mathbb{Z}^D \to \{-1,1\}$ with (quenched, formal) Hamiltonian

$$H^h(\sigma) := -\sum_{u\sim v} \sigma_u \sigma_v - \lambda \sum_v h_v \sigma_v$$

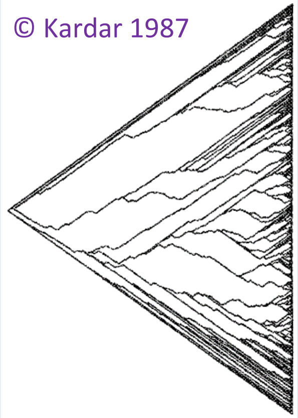
  where the (h_v) are IID N(0,1) (say), and the field strength λ is small.
- For the model of harmonic minimal surfaces, this translates to taking the disorder to be a two-sided Brownian motion (instead of regularized white noise).

Geodesics from apex to points on line

Independent disorder



Brownian disorder



Minimal surfaces in a strongly correlated random environment II

- Studying domain walls in the random-field Ising model corresponds to taking a two-sided Brownian motion disorder for harmonic minimal surfaces.
- More generally, we consider harmonic minimal surfaces with fractional Brownian disorder with Hurst parameter $H \in (0,1)$ on $\mathbb{R}^n$ (Brownian case: $H = \frac{1}{2}$ and $n = 1$).
- The disorder is not stationary but has stationary increments.
- We identify the precise height and energy fluctuation exponents in all but the critical dimension $d = 4$. The exponents are the same for all codimensions n.
- Theorem (Dembin-Elboim-P. 25): Dimensions $d = 1,2,3$: The exponents are:

$$\xi = \frac{4-d}{4-2H}, \qquad \chi = \frac{4-d}{2-H} + d - 2$$

They are determined by the two scaling relations:

$$\chi = 2\xi + d - 2 = H\xi + \frac{d}{2}$$

- Critical dimension ($d = 4$): Height delocalization in $\left[(\log\log L)^{\frac{1}{4-2H}}, (\log L)^{\frac{5}{4-2H}}\right]$.
Energy fluctuation in $[L^2(\log\log L)^{\frac{H}{4-2H}}, \mathrm{L}^2(\log \mathrm{L})^{\frac{5\mathrm{H}}{4-2\mathrm{H}}}]$.
- Dimensions $d \geq 5$: Surface is localized, with energy fluctuations of order $L^{\frac{d}{2}}$.

Brief discussion of other disorders

- Periodic disorder: $t \mapsto \eta_{v,t}$ is periodic with respect to the action of $\mathbb{Z}^n$. Stationary to $\mathbb{R}^n$ action. For $n = 1$, provides a "no vortices" approximation to the random-field XY model. Magnetization of the spin model is then in correspondence with localization of the minimal surface. Also describes random-phase Sine-Gordon.
- Our localization results hold also for suitable periodic disorders (those satisfying (stat)+(conc)).
  Thus, our proof that the $d \geq 5$ minimal surface is localized supports the prediction (still open in mathematics) that the random-field XY model retains its ferromagnetic phase at weak disorder and low temperatures in dimensions $d \geq 5$.
- Linear disorder: $\eta_{v,t} = \eta_v \cdot t$. With, e.g., each η_v distributed $N(0,1)$ (much like fractional Brownian motion with Hurst parameter $H = 1$).
  An exactly-solvable case termed random-rod, or Larkin model in physics literature.
- Height fluctuations $L^{\frac{4-d}{2}}$ for $d = 1,2,3$, $\sqrt{\log L}$ for $d = 4$ and localized for $d \geq 5$.
- Integer-valued version: Dario–Harel–P. 2023 prove localization for $d \geq 3$ at weak disorder strength λ.
  Conjecture a roughening transition as disorder strength increases for $d = 3$.

Selected open questions

- Improved exponents: For instance, for $d = 1$ is there a (large) n for which the transversal fluctuations are of order $\sqrt{L}$?
- Periodic disorder (e.g., random-phase sine-Gordon $n = 1$, Giamarchi–Le Doussal 95, Nattermann 90, Orland–Shapir 95, Villain–Fernandez 84):
  $d = 2$: Predictions of "super-roughening" (delocalization to height $\log L$).
  $d = 3$: Delocalization to height $\sqrt{\log L}$.
  Supports power-law magnetization decay prediction for $d = 3$ random-field XY model (Feldman 01, Gingras–Huse 96). What happens in dimension $d = 4$?
- Integer-valued heights (n=1): Is there a roughening transition in the disorder strength in dimension $d = 3$?
  Conjectured in Bassan–Gilboa–P. 23 for domain wall of random-bond Ising model.
  Conjectured for linear disorder in Dario–Harel–P. 23.
- Shape of the energy and fluctuation distribution:
  Prove unimodality of the distribution and concentration bounds on the scale of its standard deviation.